Indian Statistical Institute
Second Semester Examination 2004-2005
B.Math I Year
Analysis II

Time: 3 hrs | Date:13-05-05 | [Max marks : 60]

1. State Inverse function Theorem and Implicit function Theorem. [4]

2. Let $S = \{(x, y) \in R^2 : x^2 + y^2 = 1\}$ be the unit circle. Let P be a polygon of N sides with vertices on the circle S where $N \geq 3$. Assume that the polygon encloses the centre of the circle. By using Lagrange method of multipliers prove (a) and (b)

   (a) if sup{area of $P : P$ as above} is attained at P_0 then P_0 is a regular polygon

   (b) If sup{perimeter of $P : P$ as above} is attained at P_1, then P_1 is a regular polygon. [3+3 = 6]

3. Let $(X, d), (Y, m)$ be metric spaces. Let $Z = X \times Y$. Define $p : Z \times Z \to [0, \infty)$ by

$$p((x_1, y_1),\ (x_2, y_2)) = [d^2(x_1, x_2) + m^2(y_1, y_2)]^{\frac{1}{2}}$$

   Show that p is a metric. [4]

4. Let (X, d) be a metric space. Define m by

$$m(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

   for all x, y in X.

   a) Show that m is a metric on X [3]

   b) $x_n \to \alpha$ in $(X, d) \Leftrightarrow x_n \to \alpha$ in (X, m) [2]

   c) Let $F \subset X$. Show that F is a closed set in $(X, d) \Leftrightarrow F$ is a closed set in (X, m) [2]

5. Let $g : (a-\delta, \quad a+\delta) \to R$ be any continuous function such that $g(a) > g(x)$ for all $x \neq a$. Show that there exist two sequences $x_n,\ y_n$ such that $x_n < a < y_n,\ x_n \to a,\ y_n \to a$ and $g(x_n) = g(y_n)$ [3]

6. Let $f : R \to R$ be the function given by $f(x) = x + 2x^2 \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$.

   a) Show that $f'(0) = 1$ and for $x \neq 0$

   $f'(x) = 1 - 2\cos(\frac{1}{x}) + 4x\sin(\frac{1}{x})$

   $f''(x) = \sin(\frac{1}{x})[4 - \frac{2}{x^2}] - \frac{4}{x}\cos(\frac{1}{x})$ [1]

   b) Show that f' is not continuous at 0 [1]

   c) Show that there exist sequences $a_n, b_n \quad a_n > 0, \quad b_n > 0, \quad a_n \to 0, \quad b_n \to 0$ such that $f'(a_n) \to 3$ and $f''(b_n) \to -1$. [2]

   d) Show that there exist a sequence $C_n \to 0, \quad C_n > 0$ such that $f'(C_n) = 0,\ |C_n f''(C_n)| \to \infty$ [2]

   e) [By using (5) if necessary] Show that there exists two sequences $\alpha_n, \beta_n, \quad \alpha_n > 0, \beta_n > 0,\ \alpha_n < C_n < \beta_n, \quad \alpha_n \to 0, \beta_n \to 0$ such that $f(\alpha_n) = f(\beta_n)$. [2]

7. Let (X, d) be a compact metric space $f : X \to X$ any function such that $d(f(x),\ f(y)) < d(x, y)$ for all $x \neq y$. Show that f has a fixed point [3]

8. Let $g : R \to R$ be given by $g(x) = \log(1+e^x)$. Show that $|g(x) - g(y)| < |x - y|$ for all $x \neq y$ and g has no fixed point. [2]

9. a) Let $X = \bigcup_{n=3}^{\infty} [n,\ n + \frac{1}{n}]$. If $f : X \to R$ is given by $f(x) = x^2$, show that f is not uniformly continuous. [1]

   b) Let $Y = \bigcup_{n=10}^{\infty} [n,\ n + \frac{1}{n^2}]$. Define $g : Y \to R$ by $g(x) = x^2$.

   i) Let $F_N = \bigcup_{n=N}^{\infty} [n,\ n + \frac{1}{n^2}]$ for $N \geq 10$. Show that $\lim_{N\to\infty} \sup \{|g(x) - g(y)| : x, y \in F_N,\ |x - y| \leq \frac{1}{2}\} = 0$ [2]

   ii) Show that g is uniformly continuous. [3]

10. Let $a, b > 0$. Let $\underset{\sim}{g} = (g_1,\ g_2) : [0,1] \to R^2$ be any continuous function such that $\underset{\sim}{g}(0) = (0,0)$ and $\frac{g_1^2(1)}{a^2} + \frac{g_2^2(1)}{b^2} > 1$. Show that there exists p in [0,1] such that $\frac{g_1^2(p)}{a^2} + \frac{g_2^2(p)}{b^2} = 1$ [3]

11. a)Show that (X, d) is disconnected $\Leftrightarrow$ there exists a continuous onto function $f : X \to \{0, 1\}$. [3]

    b) Show that continuous image of a connected set is connected. [2]

    c) If $A_1, A_2, \ldots, A_k$ are connected subsets of (X, d) such that $A_i \cap A_{i+1}, \neq$ empty for $i = 1, 2, \ldots k-1$, then show that $A_1 \cup A_2 \cup \ldots \cup A_k$ is connected [3]

    d) Let

$$\begin{aligned} A &= \{(x, y) : \quad x^2 + y^2 = 1\} \\ B &= \{(x, y) : \quad x^2 + y^2 = 2\} \\ C &= \{(x, y) : y = 0\}. \end{aligned}$$

    Show that each of the sets $A, B, C, A \cup B \cup C$ is connected. [4]

12. Let $A = \{x \in R : x \text{ is rational}, 0 \leq x \leq 2\}$, show that

    a) A is closed subset of rationals.

    b) A is not a closed subset of reals.

    c) A is not compact. [3]